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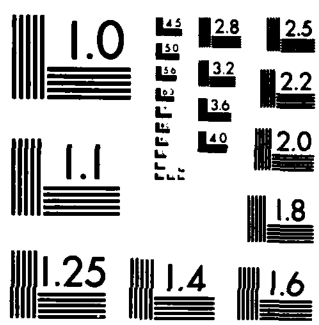


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Laboratory for Numerical Analysis

Technical Note BN-1005

# IDENTIFICATION OF AN UNKNOWN CONDUCTIVITY BY MEANS OF MEASUREMENTS AT THE BOUNDARY



By

Robert V. Kohn and Michael Vogelius





June 1983



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We present a summary of results concerning the determination of an unknown conductivity by means of static measurements at the boundary. The main emphasis is on identifiability; we only briefly discuss the reconstruction problem. Some references are given to related work for time dependent problems.

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# IDENTIFICATION OF AN UNKNOWN CONDUCTIVITY BY MEANS OF MEASUREMENTS AT THE BOUNDARY

Robert V. Kohn*

and

Michael Vogelius*

ABSTRACT. ~~We~~ The authors present a summary of results concerning the determination of an unknown conductivity by means of static measurements at the boundary. The main emphasis is on identifiability; ~~we~~ they only briefly discuss the reconstruction problem. Some references are given to related work for time dependent problems.

1. INTRODUCTION. We study the following inverse problem: can one determine an unknown conductivity $\gamma(x)$ inside a body $\Omega$, by means of static measurements at the boundary? A. P. Calderón raised this question in [2]; it may be seen as a natural extension of a problem analyzed by Cannon, Douglas and Jones in 1963 [3,4]. Despite some progress, many unsolved problems remain.

Our main goal here is to summarize what is known about identifiability; we do this in sections 2 and 3, which are based mostly on [18]. A few of the results presented - notably 2E and 3C - are previously unpublished. In section 4, we touch on the reconstruction problem, given finitely many measurements; and section 5 reviews the literature on some related problems. Potential applications include nondestructive testing and water resources management [14,13], but these will not be discussed here.

Throughout, $\Omega$ will be a bounded domain in $\mathbb{R}^n$, with unit normal $\nu$ along $\partial\Omega$. The unknown conductivity $\gamma(x)$ may take real scalar or matrix values, corresponding to an isotropic or anisotropic material. In the isotropic case

$$(1.1) \qquad \gamma \in L^\infty(\Omega)\ , \ \text{ess inf}\ \gamma > 0 \ ,$$

while in the anisotropic case

$$(1.2) \qquad \gamma_{ij} = \gamma_{ji} \in L^\infty(\Omega)\ , \quad \lambda|\xi|^2 \le (\gamma(x)\xi,\xi) \le \Lambda|\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, with $\lambda > 0$. We consider solutions of

$$(1.3) \qquad L_\gamma u = \nabla\cdot(\gamma(x)\nabla u) = 0 \ ;$$

in the context of heat conduction $u$ represents temperature and $\gamma\nabla u$ the heat

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flux; in the context of direct current electrical conduction $u$ represents voltage and $\gamma\nabla u$ is the vector of current flow. The natural things to measure at $\partial\Omega$ are the Dirichlet data $u|_{\partial\Omega}$ and the Neumann data $\gamma(x)\nabla u\cdot\nu|_{\partial\Omega}$. We denote by $P_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ the operator which associates the former to the latter,

$$P_\gamma\phi = \gamma\nabla u\cdot\nu|_{\partial\Omega} \text{ with } L_\gamma u = 0 \ , \ u|_{\partial\Omega} = \phi \ ;$$

we shall say that "$\gamma_1$ and $\gamma_2$ give the same boundary measurements" if $P_{\gamma_1} = P_{\gamma_2}$.

Knowledge of $P_\gamma$ yields the energy quadratic form

$$Q_\gamma(\phi) = \int_\Omega (\gamma\nabla u, \nabla u)dx = \int_{\partial\Omega} \phi\cdot P_\gamma\phi ds \ , \tag{1.4}$$

by Green's formula, where the rightmost integral represents the dual pairing of $H^{1/2}$ and $H^{-1/2}$. Conversely, $Q_\gamma$ determines $P_\gamma$ by the polarization identity; hence "$\gamma_1$ and $\gamma_2$ give the same boundary measurements" iff $Q_{\gamma_1}(\phi) = Q_{\gamma_2}(\phi)$ for each $\phi \in H^{1/2}(\partial\Omega)$. The following variational characterizations of $Q_\gamma$ are well-known:

$$Q_\gamma(\phi) = \min_{\substack{w \in H^1(\Omega) \\ w=\phi \text{ on } \partial\Omega}} \int_\Omega (\gamma\nabla w, \nabla w)dx \tag{1.5}$$

$$-Q_\gamma(\phi) = \min_{\substack{\sigma\in L^2(\Omega;\mathbb{R}^n) \\ \nabla\cdot\sigma=0}} \left( \int_\Omega (\gamma^{-1}\sigma,\sigma)dx - 2\int_{\partial\Omega} \phi\sigma\cdot\nu ds \right) \ . \tag{1.6}$$

2. IDENTIFIABILITY - THE ISOTROPIC CASE. In one dimension, only the harmonic mean of $\gamma$ can be detected by boundary measurement: it is an easy exercise to show that

2A. <u>For $\Omega = (a,b)$, $\gamma_1$ and $\gamma_2$ give the same boundary measurements iff they have the same harmonic mean.</u>

Fortunately, the situation is entirely different in dimension greater than one.

Cannon, Douglas, and Jones considered cylindrical domains, with $\gamma$ constant along lines parallel to the axis, in 1963. They showed that such $\gamma$ are identifiable:

2B [3]. <u>Suppose $\Omega = G \times (a,b)$, with $G$ a bounded, $C^{2,\alpha}$ domain in $\mathbb{R}^{n-1}$, $\alpha > 0$. Then the elements of</u>

$$\Gamma_1 = \{\gamma \in C^{1,\alpha}(\bar{G}) : \inf \gamma > 0\}$$

<u>can be distinguished by means of boundary measurements.</u>

Their procedure for reconstructing $\gamma$ is remarkably direct. Taking $(a,b) = (0,\pi)$ for simplicity, and writing $x' = (x_1,\ldots,x_{n-1})$, let $L_\gamma u = 0$ with

$$u = 0 \qquad \text{on} \quad G \times \{0,\pi\}$$

$$u = g(x')\sin x_n \quad \text{on} \quad \partial G \times (0,\pi) ,$$

where $g : \partial\Omega \to \mathbb{R}$ is any positive function. If

$$k(x') = \gamma(x') \left.\frac{\partial u}{\partial x_n}\right|_{x_n=0}$$

is measured, then

$$\gamma = k \cdot \exp(-w) ,$$

where $w(x')$ solves

$$\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\left(k \frac{\partial w}{\partial x_j}\right) = k \qquad \text{on} \quad G$$

$$w = \ln g \quad \text{on} \quad \partial G$$

Thus $\gamma(x')$ is determined for <u>all</u> $x'$, using a <u>single</u> choice of $g$.

The restriction that $\gamma$ be independent of $x_n$ is, of course, crucial to the preceeding analysis. One is not really finding $\gamma$ "in the interior", since it is determined by its values along the boundary. When the dependence of $\gamma$ is unrestricted, one naturally obtains information at the boundary more easily than in the interior. If everything is smooth, then $\gamma$ is determined to infinite order at $\partial\Omega$.

<u>2C</u> [18]. <u>Suppose that</u> $\partial\Omega$ <u>is smooth, and that</u> $\gamma_1, \gamma_2 \in C^\infty(\bar\Omega)$. <u>If</u> $\gamma_1$ <u>and</u> $\gamma_2$ <u>give the same boundary measurements, then</u>

$$D^{\underline{k}}\gamma_1 = D^{\underline{k}}\gamma_2 \quad \underline{\text{on}} \quad \partial\Omega \tag{2.1}$$

<u>for all</u> $\underline{k} = (k_1,\ldots,k_n) \geq 0$, <u>where</u> $D^{\underline{k}} = \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n}$.

The proof of 2C is local in character, but not constructive. For $x_0 \in \partial\Omega$ with $\nu_n(x_0) \neq 0$, consider the Dirichlet data

$$\phi_N(x) = N^{\frac{n}{2}-1} \prod_{j=1}^{n-1} \psi\,(N(x_j - x_{0,j}))$$

with corresponding solutions $u_N^i$,

$$L_{\gamma_i} u_N^i = 0 , \quad \left.u_N^i\right|_{\partial\Omega} = \phi_N , \quad i = 1,2 .$$

If $\psi \in C_0^\infty(\mathbb{R})$ has vanishing moments of order $\leq M-1$, then a version of "St. Venant's principle" provides that

$$|\nabla u_N^i(x)| \leq C\, N^{-M} , \quad N \to \infty$$

for $x \in \bar\Omega$ bounded away from $x_0$.

If (2.1) fails, then (relabeling if necessary)

$$\gamma_1(x) - \gamma_2(x) > C \text{ dist } (x,\partial\Omega)^{\ell}$$

with $\ell \geq 0$ and $C > 0$, in a neighborhood of some $x_0 \in \partial\Omega$ ; it now follows that $Q_{\gamma_1}(\phi_N) > Q_{\gamma_2}(\phi_N)$ provided $M > n\ell/2$ and $N$ is sufficiently large, a contradiction to the fact that $\gamma_1$ and $\gamma_2$ give the same boundary measurements. Details may be found in [18]; see also 3C below.

An *immediate* corollary of 2C is this:

<u>2D</u> : <u>For a smoothly bounded domain</u> $\Omega \subset \mathbb{R}^n$ , $n \geq 2$ , <u>the elements of</u>

$$\Gamma_2 = \{\text{restrictions to } \Omega \text{ of positive, real analytic functions defined in a neighborhood of } \bar{\Omega}\}$$

<u>can be distinguished by boundary measurements</u>.

The analogue of 2D with "analytic" replaced by "piecewise analytic" is open. We are encouraged, however, by the following example.

<u>2E</u>. <u>Let</u> $\Omega$ <u>be the unit disc in</u> $\mathbb{R}^2$ , <u>with polar coordinates</u> $(r,\theta)$ , <u>and denote by</u> $\Gamma_3$ <u>the set of conductivities</u>

$$\gamma(r,\theta) = \begin{cases} \gamma_0 & r < r_0 \\ 1 & r_0 \leq r < 1 \end{cases}$$

<u>with</u> $0 < r_0 < 1$ <u>and</u> $\gamma_0$ <u>a positive constant</u>. <u>Then the elements of</u> $\Gamma_3$ <u>can be distinguished by boundary measurements</u>.

To prove 2E , consider $\gamma, \tilde{\gamma} \in \Gamma_3$ , with $\tilde{\gamma}$ corresponding to $\tilde{\gamma}_0$ , $\tilde{r}_0$ ; we shall show that $Q_\gamma(\sin N\theta) \neq Q_{\tilde{\gamma}}(\sin N\theta)$ for all sufficiently large $N$ , unless $\tilde{\gamma} = \gamma$ . If $\tilde{r}_0 = r_0$ this follows instantly, since either $\tilde{\gamma} < \gamma$ or $\tilde{\gamma} > \gamma$ and consequently either $Q_{\tilde{\gamma}}(\sin N\theta) < Q_\gamma(\sin N\theta)$ or $Q_{\tilde{\gamma}}(\sin N\theta) > Q_\gamma(\sin N\theta)$ , provided $\tilde{\gamma}_0 \neq \gamma_0$ .

Relabeling if necessary, we assume that $\tilde{r}_0 < r_0$ . For $N \geq 0$ , let $u_N$ solve

$$L_\gamma u_N = 0 \quad , \quad u_N\big|_{\partial\Omega} = \sin N\theta \quad ,$$

and notice that for $r < r_0$

$$(2.2) \qquad u_N = c_N r^N \sin N\theta \quad , \quad c_N = \frac{2}{r_0^{2N}(1-\gamma_0)+(\gamma_0+1)} \; .$$

In case $\gamma_0 > 1$ , (2.2) implies that

$$(2.3) \qquad \int_{r<\tilde{r}_0} \tilde{\gamma}_0 |\nabla u_N|^2 dx < \int_{\tilde{r}_0<r<r_0} (\gamma_0-1)|\nabla u_N|^2 dx$$

for sufficiently large N, whence

$$\int_\Omega \tilde{\gamma}|\nabla u_N|^2 dx = \int_{r<\tilde{r}_0} \tilde{\gamma}_0|\nabla u_N|^2 dx + \int_{\tilde{r}_0<r<r_0} |\nabla u_N|^2 dx + \int_{r_0<r<1} |\nabla u_N|^2 dx$$

$$< \int_{\tilde{r}_0<r<r_0} \gamma_0|\nabla u_N|^2 dx + \int_{r_0<r<1} |\nabla u_N|^2 dx ,$$

so

$$\int_\Omega \tilde{\gamma}|\nabla u_N|^2 dx < \int_\Omega \gamma|\nabla u_N|^2 dx = Q_\gamma(\sin N\theta) . \tag{2.4}$$

It follows, using (1.5), that $Q_{\tilde{\gamma}}(\sin N\theta) < Q_\gamma(\sin N\theta)$ .

Next, suppose that $\gamma_0 < 1$ . In this case, a similar argument gives

$$\int_\Omega \tilde{\gamma}^{-1}|\sigma_N|^2 dx < \int_\Omega \gamma^{-1}|\sigma_N|^2 dx \tag{2.5}$$

with $\sigma_N = \gamma\nabla u_N$ , and $N$ sufficiently large. Using the dual variational principle (1.6), we conclude that $Q_{\tilde{\gamma}}(\sin N\theta) > Q_\gamma(\sin N\theta)$ .

3. IDENTIFIABILITY - THE ANISOTROPIC CASE. In the *anisotropic case*, one can not expect to recover the full matrix $\gamma_{ij}$ , as the following two examples demonstrate.

3A. Let $\Omega \subset \mathbb{R}^n$ , $n \geq 1$ , and let $\gamma$ satisfy (1.2). For any $C^1$ diffeomorphism $\Phi : \Omega \to \Omega$ with

$$\Phi(x) = x , \quad D\Phi(x) = I \ \text{for all} \ x \in \partial\Omega , \tag{3.1}$$

let

$$\gamma^\Phi(\Phi(x)) = |\det(D\Phi(x))|^{-1} \cdot D\Phi(x)^t \cdot \gamma(x) \cdot D\Phi(x) .$$

Then all elements of

$$\Gamma_4 = \{\gamma^\Phi : \Phi \ \text{satisfies (3.1)}\}$$

give the same boundary measurements.

We owe this remark to L. Tartar. If $L_\gamma u = 0$ , then $L_{(\gamma^\Phi)} u^\Phi = 0$ , with

$$u^\Phi(x) = u \circ \Phi^{-1}(x) ;$$

by (3.1), $u^\Phi = u$ on $\gamma^\Phi \cdot \nabla u^\Phi = \gamma \cdot \nabla u$ on $\partial\Omega$ .

3B [25]. Let $\Omega$ be the unit disc in $\mathbb{R}^2$ , with polar coordinates $(r,\theta)$ . For any function $\alpha(r)$ , let

$$\gamma^\alpha = \begin{pmatrix} \alpha\cos^2\theta + \alpha^{-1}\sin^2\theta & (\alpha-\alpha^{-1})\sin\theta\cdot\cos\theta \\ (\alpha-\alpha^{-1})\sin\theta\cdot\cos\theta & \alpha\sin^2\theta + \alpha^{-1}\cos^2\theta \end{pmatrix} .$$

Then all elements of

$$\Gamma_5 = \{\gamma^\alpha : \alpha \in L^\infty(0,1) , \ \text{ess inf}\ \alpha > 0\}$$

give the same boundary measurements.

Indeed,

$$r\cdot L_{\gamma^\alpha} = \frac{\partial}{\partial r}\left(r\alpha(r)\frac{\partial}{\partial r}\right) + \frac{1}{r\alpha(r)}\frac{\partial^2}{\partial\theta^2} ;$$

note that when $\alpha \equiv 1$ this is just $r$ times the Laplacian. For $N \in Z$, the solution of

$$L_{\gamma^\alpha} u_N = 0 , \quad u_N\big|_{\partial\Omega} = e^{iN\theta}$$

has the form $v(r)e^{iN\theta}$, with $v(1) = 1$ and

$$\frac{\partial}{\partial r}\left(r\alpha(r)\frac{\partial v}{\partial r}\right) - \frac{N^2}{r\alpha(r)} v = 0 .$$

This implies that

$$v(r) = c_1 \exp\left(|N|\int_1^r \frac{ds}{s\alpha(s)}\right) + c_2 \exp\left(-|N|\int_1^r \frac{ds}{s\alpha(s)}\right) ,$$

with $c_1 + c_2 = 1$. Since $v(r)e^{iN\theta} \in H^1(\Omega)$, $c_2$ must equal zero; hence the Neumann data associated to $u_N$ is

$$\gamma^\alpha \nabla u_N\cdot\nu\Big|_{r=1} = \alpha\frac{\partial u_N}{\partial r} = |N|e^{iN\theta} ,$$

regardless of the choice of $\alpha$. The span of $\{e^{iN\theta}\}$ is dense in $H^{1/2}(\partial\Omega)$, so each $\gamma^\alpha \in \Gamma_5$ gives the same boundary measurements.

What <u>can</u> one detect, in the anisotropic case? The natural analogue of 2C is this: if (n-1) eigenvalues and eigenvectors of $\gamma$ are known, the last eigenvalue can be distinguished by *boundary measurements*.

<u>3C. Let $\gamma,\tilde\gamma$ be two symmetric, positive definite matrices with entries in $L^\infty(\Omega)$, and let $\{\lambda_i\},\{\tilde\lambda_i\}$ and $\{e_i\},\{\tilde e_i\}$ be the corresponding eigenvalues and eigenvectors. For $x_0 \in \partial\Omega$, let $B$ be a neighborhood of $x_0$ relative to $\bar\Omega$, and suppose that</u>

(3.2) $$\gamma,\tilde\gamma \in C^\infty(B) , \text{ and } \partial\Omega \cap B \text{ is } C^\infty ;$$

(3.3) $$e_j = \tilde e_j , \quad \lambda_j = \tilde\lambda_j \text{ in } B , \quad 1 \le j \le n-1 ;$$

(3.4) $$e_n(x_0)\cdot\nu(x_0) \ne 0$$

(3.5) $$Q_\gamma(\phi) = Q_{\tilde\gamma}(\phi) \text{ for every } \phi \in H^{1/2}(\partial\Omega) \text{ with } \operatorname{supp}\phi \subset B \cap \partial\Omega .$$

<u>Then</u>

(3.6) $$D^{\underline{k}}\lambda_n(x_0) = D^{\underline{k}}\tilde\lambda_n(x_0)$$

<u>for any</u> $\underline{k} = (k_1,\ldots,k_n) \ge 0$.

We sketch the proof. For a fixed $z \in \partial\Omega$ near $x_0$, let $\{\phi_N\}_{N=1}^{\infty}$ be a sequence of functions on $\partial\Omega$ such that

$$\text{supp}\ \phi_N \downarrow \{z\} \tag{3.7}$$

$$\|\phi_N\|_{k,\partial\Omega\cap B} \leq C_k N^k \text{, for all } k \geq -M \tag{3.8}$$

$$\|\phi_N\|_{0,\partial\Omega\cap B} = 1 , \tag{3.9}$$

where the norms are standard Sobolev norms based on $L^2$; the existence of such a sequence may be deduced from [18], for any fixed $M > 0$. If $u_N$ solves

$$L_\gamma u_N = 0 \text{ in } \Omega , \quad u_N = \phi_N \text{ on } \partial\Omega ,$$

then (3.7)-(3.9) imply

$$\|u_N\|_{1,\Omega\setminus U} \leq CN^{-M} \tag{3.10}$$

for all neighborhoods $U$ of $z$ (c.f. Lemma 2 of [18]). Condition (3.4) gives in a sufficiently small neighborhood $U$ of $z$

$$\int_{U\cap\partial\Omega} |w|^2 ds \leq C \int_U |e_n\cdot\nabla w|^2 dx \tag{3.11}$$

for any $w \in H^1(\Omega)$ with $w = 0$ in $\Omega\setminus U$; using this and (3.7)-(3.9), one obtains

$$\int_U \rho^\ell |e_n\cdot\nabla u_N|^2 dx \geq C_{\ell,\varepsilon} N^{-(n+1+\varepsilon)\ell} , \quad C_{\ell,\varepsilon} > 0 \tag{3.12}$$

for any $\ell \geq 0$, $\varepsilon > 0$, with $\rho(x) = \text{dist}(x,\partial\Omega)$, provided $M > (n+1)\ell/2$ (cf. Lemma 3 of [18]).

In order to prove (3.6) it is sufficient to verify that $\left(\frac{\partial}{\partial\nu}\right)^k\gamma = \left(\frac{\partial}{\partial\nu}\right)^k\tilde{\gamma}$ in a $\partial\Omega$ - neighborhood of $x_0$, for any $k \geq 0$. We prove this by contradiction, using (3.10) and (3.12). If it fails, we may assume (switching $\gamma$ and $\tilde{\gamma}$ if necessary) that there exists a $z$ near $x_0$ and a neighborhood $U \subseteq B$ of $z$ such that

$$\lambda_n(x) - \tilde{\lambda}_n(x) \geq C\rho(x)^\ell \text{ for } x \in U , \tag{3.13}$$

with $C > 0$, $\ell \geq 0$. Then

$$\int_\Omega (\gamma\nabla u_N, \nabla u_N)dx \geq \int_U (\gamma\nabla u_N, \nabla u_N)dx \geq \int_U (\tilde{\gamma}\nabla u_N, \nabla u_N)dx$$
$$+ C\int_U \rho^\ell |e_n\cdot\nabla u_N|^2 dx ,$$

using (3.3) and (3.13). If $M > (n+1)\ell/2$, then (3.10) and (3.12) show that

$$C\int_U \rho^\ell |e_n\cdot\nabla u_N|^2 dx > \int_{\Omega\setminus U} (\tilde{\gamma}\nabla u_N, \nabla u_N)dx$$

for large N. Hence, using (1.5),

$$Q_\gamma(\phi_N) = \int_\Omega (\gamma\nabla u_N, \nabla u_N)dx > \int_\Omega (\tilde{\gamma}\nabla u_N, \nabla u_N)dx \geq Q_{\tilde{\gamma}}(\phi_N) ;$$

This contradicts (3.5), and the proof is complete.

The preceding argument used (3.4) to justify (3.11). The following example suggests that (3.4) may be dispensable.

3D [4]. *For* $\Omega = (a,b) \times (c,d) \subset \mathbb{R}^2$ , *let* $\Gamma_6$ *denote the family of conductivities*

$$\gamma(x) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha(x_2) \end{pmatrix},$$

*for* $\alpha \in C^1$ *with* $\inf \alpha > 0$ . *Then the elements of* $\Gamma_6$ *can be distinguished by boundary measurements. More specifically, if* $\gamma_i \in \Gamma_6$ , $i = 1,2$ *and*

$$\gamma_1 \nabla u^1 \cdot \nu = \gamma_2 \nabla u^2 \cdot \nu \quad \text{at} \quad x_1 = b$$

*where* $u^i$ *is the solution to*

$$L_{\gamma_i} u^i = 0 \quad \text{in} \quad \Omega$$

$$u^i = 0 \quad \text{at} \quad x_1 = a,b$$

$$u^i = \sin\left(\pi \frac{x_1 - a}{b-a}\right) \quad \text{at} \quad x_2 = c,d$$

*then*

$$\gamma_1 \equiv \gamma_2 .$$

4. RECONSTRUCTION. Setting aside the question of identifiability, how might one estimate $\gamma$ in practice? A straightforward approach is the following: let $V$ be a finite-dimensional subspace of $H^1(\Omega)$ ; for $w_j \in V$, $1 \le j \le m$ , set $\phi_j = w_j|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ , and measure $\gamma \frac{\partial u_j}{\partial \nu}$ , where $u_j$ solves

$$L_\gamma u_j = 0 \quad \text{in} \quad \Omega , \quad u_j|_{\partial\Omega} = \phi_j .$$

If $G$ is a finite-parameter family of possible conductivities, and $\tilde{\gamma} \in G$ , let $\tilde{u}_j \in V$ be the solution to the Galerkin equation

$$\int_\Omega (\tilde{\gamma}\nabla\tilde{u}_j, \nabla\phi)dx = 0 \quad \text{for all} \quad \phi \in V \cap \mathring{H}^1(\Omega)$$

$$\tilde{u}_j = \phi_j \quad \text{on} \quad \partial\Omega .$$

Now assume furthermore that $V$ and $G$ are selected so that $\tilde{\gamma} \frac{\partial w}{\partial \nu} \in H^{-1/2}(\partial\Omega)$ whenever $w \in V$ and $\tilde{\gamma} \in G$ , and choose $\tilde{\gamma}$ to minimize

$$J(\tilde{\gamma}) = \sum_{j=1}^{m} \left\| \tilde{\gamma}\frac{\partial \tilde{u}_j}{\partial \nu} - \gamma \frac{\partial u_j}{\partial \nu} \right\|^2_{H^{-1/2}(\partial\Omega)} \tag{4.1}$$

among all $\tilde{\gamma} \in G$ .

A variation of this method, and its finite-difference analogue, is studied by Falk in [11], for operators of the form

$$- \gamma u'' + c(x)u = f(x) \qquad 0 < x < 1$$

$$c,f \ \text{ known; } \ \gamma > 0 \ \text{ constant}$$

in one dimension. He takes $G = \mathbb{R}_+$ , and uses just one measurement ($m=1$) at one point on the boundary ($x=0$), so that the *functional* $= 0$ *for some* $\tilde{\gamma}$ ; and he estimates how $|\tilde{\gamma}-\gamma|$ depends on the choice of $V$ .

The minimization of (4.1) has apparently not been studied in higher dimensions. There is, however, a large literature on parameter identification, much of which is relevant: see, for example, [10,19,22]. In other problems where identificability and well-posedness are known, one can often prove the convergence of such a procedure [1]. And in some cases, even identifiability can be proved using an approximation algorithm [20].

Calderón takes a completely different approach to the reconstruction problem in [2], for the scalar case with $\delta = \|\gamma-1\|_{L^\infty}$ sufficiently small. He shows that the Fourier transform of $\gamma$ (extended by zero off $\Omega$) has the form

$$\hat{\gamma}(\xi) = f(\xi) + R(\xi) \ ,$$

where $f(\xi)$ can be determined by boundary measurements, and

$$|R(\xi)| \leq C\cdot\delta^2\cdot\exp(\pi\cdot|\xi|\cdot\text{diam}(\Omega)) \ .$$

Thus boundary measurements suffice to *approximate* $\hat{\gamma}(\xi)$ well at low frequencies, if $\delta$ is small. For any fixed $\xi \in \mathbb{R}^n$ , measuring $f(\xi)$ requires the Neumann data from just one solution of $L_\gamma u = 0$ , corresponding to Dirichlet conditions $\exp[\pi(i\xi+\eta)\cdot x]$ , with $\eta \in \mathbb{R}^n$ , $\eta\cdot\xi = 0$ , $|\eta| = |\xi|$ .

5. RELATED WORK. The parabolic analogue of our problem is to determine $\gamma = \gamma(x,t)$ , given overdetermined boundary data for solutions of

$$\frac{\partial u}{\partial t} - \nabla\cdot(\gamma\nabla u) = 0$$

on a space-time cylinder $\Omega \times (0,T)$ . This problem has been studied extensively in space-dimension one, both as to the identifiability of $\gamma$ and as to its numerical approximation: see [6] for $\gamma$ = constant, [5,9,15,16] for $\gamma = \gamma(t)$ , and [17,21,26] for $\gamma = \gamma(x)$ . We know of no results in space dimension greater than one for *spatially* varying $\gamma$ .

An interesting nonlinear analogue is obtained by letting $\gamma$ depend on $u$ , so that the equation becomes $\nabla\cdot(\gamma(x,u)\nabla u) = 0$ . The case $\gamma = \gamma(u)$ has been studied in [7], and a related parabolic problem is treated in [8].

Many authors have studied the reconstruction of an unknown $\gamma(x)$ , given knowledge of a single function $u$ everywhere on $\Omega$ , satisfying $\nabla\cdot(\gamma\nabla u) = 0$ . This problem is of particular interest for studying ground-water flow through

porous rock. Identifiability is analyzed in [24], and the convergence of numerical schemes are studied in [23,12]. Applications and other numerical methods have been discussed in a dozen or so articles in Water Resources Research over the last ten years, of which [13,27] are examples.

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The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

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